

Algebraic on Magic Square of Odd Order n

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Abstract. This paper aims to address the relation between a magic square of odd order n and a group, and their properties. By the modulo number n , we construct entries for each table from initial table of magic square with large number n^2 . Generalization of the underlying idea is presented, we obtain unique group, and we also prove variants of the main results for magic cubes.

Keywords: entry, array, algorithm, magic cubes, group

1 Introduction

According to the book of W. S. Andrews [1], the study of magic squares is quite old and dates back to ancient Tibet, to 12th century China, to 9th century Arab astrologers and perhaps much further. Speculation about it might even be prehistoric.

In this paper, we shall see old procedure can product unique magic square based on a group of a set f numbers in modulo n . Objectives are find new magic square which it has different procedure if it compares with old. Therefore, we get a procedure for generating new magic square. Of course, this paper organized by first we defined the magic square with a condition for producting it, and based on conditions we result a procedure on odd order n . Next section, we modify all entries of magic square on modulo n and we test all conditions of magic square. In finding new procedure for the magic square with all entries on modulo n , we force a simple group on a set Z_n with a binary operation, and based on we find new magic square. What are all magic squares of odd order n satisfying the condition of magic square?

2 The Magic Squares

A magic square (MS) of order n , also called a $n \times n$ magic square, or it means that $n \times n$ square array $A = (a_{ij})$ $0 \leq, j \leq n - 1$, of positive integers such that

- each integer from 1 to n^2 inclusive occurs exactly once among the entries of A ,
- for $0 \leq i \leq n - 1$, the sum $\sum_{j=0}^{n-1} a_{ij}$ is independent of i ,

- c. for $0 \leq i \leq n-1$, the sum $\sum_{j=0}^{n-1} a_{ij}$ is independent of j ,
d. the sums $\sum_{i=0}^{n-1} a_{ii}$ and $\sum_{i=0}^{n-1} a_{i,n-i-1}$ are equal to the sums given in (b.), and such as to those in (c.).

8	1	6
3	5	7
4	9	2

Fig. 1. 3×3 magic square (of odd order).

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Fig. 1. 4×4 magic square (of even order).

Let A be the 3×3 magic square for example in Fig. 1, and B be the 4×4 magic square in Fig. 2. The product square $A * B$ has order 12, the product of the orders of A and B . Let G be an abelian group and choose an element u of G once and for all. Denote by $A(G)$ the set of all square arrays of elements of G , the size of the arrays being arbitrary. If $A = (a_{ij})$, $0 \leq i, j \leq m-1$, and $B = (b_{kl})$, $0 \leq k, l \leq n-1$ are elements of $A(G)$ of size $m \times m$ and $n \times n$ respectively, then $A * B$ will be the $mn \times mn$ matrix $E = (e_{\alpha\beta})$, (α, β) -the entry.

A set S closed under an associative operation is called a semigroup, and the set be monoid if its operation with an identity element. The following are some results [2].

Lemma 1. *Let G be an abelian group. The set $A(G)$ of all square arrays with entries in G is a monoid with identity element u with respect to the operation $*$ defined by*

$$e_{\alpha\beta} = m^2(b_{kl} + u) + a_{ij} \quad (1)$$

and

$$(\alpha, \beta) = m(k, l) + (i, j) \quad (2)$$

Let S be a monoid with identity element of S that respect to an operation. We say that S is left cancellative for the other side. In this case, the entries of a magic square of order n run from 0 to $n^2 - 1$, and we take $u = 0$ bukan $u = -1$.

For A is $m \times m$, B is $n \times n$ and C is $p \times p$. If $A * C = B * C$ then we must have $mp = np$ and thus $m = n$. Therefore, we have $m^2(c_{kl} - 1) + a_{ij} = m^2(c_{kl} - 1) + b_{ij}$ for $0 \leq i, j \leq m-1$ and $0 \leq k, l \leq n-1$, which implies that $a_{ij} = b_{ij}$ for all i, j . That means that $A * B = B * C$, it implies $A = B$. If $A * B = A * C$ then we must have $mn = mp$ and therefore $n = p$. It follows that $m^2(b_{kl} + 1) + a_{ij} = m^2(c_{kl} - 1) + a_{ij}$ for $0 \leq i, j \leq m-1$ and $0 \leq k, l \leq n-1$, which implies that $m^2(b_{kl} - c_{kl}) = 0$ for all k and l . It follows that $b_{kl} = c_{kl}$ for all k, l , and equation $A * B = A * C$ implies $B = C$.

Lemma 2. *Let G be an abelian group and let u be an element of G . Then the monoid $(A(G), *, u)$ is right cancellative.*

Lemma 3. *Let G be an abelian group and let u be an element of G . Then the monoid $(A(G), *, u)$ is left cancellative if and only if the group G is torsion-free.*

3 The Odd Order

Let we consider the magic square of odd order n , $n = 3, 5, 7, 9, \dots$. All condition (a.) - (b.) above give entries of a magic square of odd order n based on following steps:

1. Set $i = (n - 1)/2$, $j = 0$, and $k = 1$.
2. Do while $k \leq n^2$
 - (a) If $j = -1$ Then
 - i. If $i = n$ Then $i = i - 1$ and $j = j + 2$ Else $j = n - 1$
 - (b) If $i = n$ Then $i = 0$
 - (c) If $a_{ij} \geq 0$ Then $j = j + 2$ and $i = i - 1$
 - (d) $a_{ij} = k$
 - (e) $j = j - 1$, $i = i + 1$ and $k = k + 1$
3. End Do

For example, we obtain the sequential numbers from lines of $n \times n$ magic squares:

$n = 5$, it is 17, 24, 1, 8, 15, 23, 5, 7, 14, 16, 4, 6, 13, 20, 22, 10, 12, 19, 21, 3, 11, 18, 25, 2, 9;

$n = 7$, it is 30, 39, 48, 1, 10, 19, 28, 38, 47, 7, 9, 18, 27, 29, 46, 6, 8, 17, 26, 35, 37, 5, 14, 16, 25, 34, 36, 45, 13, 15, 24, 33, 42, 44, 4, 21, 23, 32, 41, 43, 3, 12, 22, 31, 40, 49, 2, 11, 20; and

$n = 9$, it is 47, 58, 69, 80, 1, 12, 23, 34, 45, 57, 68, 7, 78, 9, 11, 22, 33, 44, 46, 67, 78, 8, 10, 21, 32, 43, 54, 56, 77, 7, 18, 20, 31, 42, 53, 55, 66, 6, 17, 19, 30, 41, 52, 63, 65, 76, 16, 27, 29, 40, 51, 62, 64, 75, 5, 26, 28, 39, 50, 61, 72, 74, 4, 15, 36, 38, 49, 60, 71, 73, 3, 14, 25, 37, 48, 59, 70, 81, 2, 13, 24, 35;

are magic squares of odd order $n = 5, 7, 9$.

If we modify condition (a.) of magic square so that the entries of a magic square of odd order n run from 1 to n^2 modulo n instead of from 1 to n^2 , we get a difference between a column (left side) and next column (to right side) is 2 in modulo n and a difference between a row and next row from top to down is 1 in modulo n . Therefore, we define function $f : Z_n \times Z_n \rightarrow Z_n$ or a binary operation on Z_n defined by $f(r, c) = 2 + r + 2c$, for all $r, c \in Z_n$, r and c represent row and column of magic square, respectively. For example, $n = 3, 5, 7$ and 9, we have new squares likes Figs. 3, 4, 5 and 6. Each of this squares in Z_n with f , $n = 3k$, $k = 1, 3, 5, 7, 9, \dots$. We obtain result of condition (d.) of a magic square, $\sum_{i=0}^{n-1}$ (we called it as sltdd or *sums of left-top-down-diagonal*) is not equal to $\sum_{i=0}^{n-1}$ (we called it as slbud or *sums of left-bottom-up-diagonal*) or other conditions (sr = sums of row and sc = sums of column), see Table A in Appendix.

2	1	0
0	2	1
1	0	2

Fig. 3. 3×3 magic square in modulo 3.

2	4	1	3	0
3	0	2	4	1
4	1	3	0	2
0	2	4	1	3
1	3	0	2	4

Fig. 4. 5×5 magic square in modulo 5.

2	4	6	1	3	5	0
3	5	0	2	4	6	1
4	6	1	3	5	0	2
5	0	2	4	6	1	3
6	1	3	5	0	2	4
0	2	4	6	1	3	5
1	3	5	0	2	4	6

Fig. 5. 7×7 magic square in modulo 7.

2	4	6	8	1	3	5	7	0
3	5	7	0	2	4	6	8	1
4	6	8	1	3	5	7	0	2
5	7	0	2	4	6	8	1	3
6	8	1	3	5	7	0	2	4
7	0	2	4	6	8	1	3	5
8	1	3	5	0	0	2	4	6
0	2	4	6	8	1	3	5	7
1	3	5	0	0	2	4	6	8

Fig. 6. 9×9 magic square in modulo 9.

A set Z_n with a binary operation f is not an associative operation, and then (Z_n, f) is not a group. Let us have a procedure to product new magic square (it is called New MS), may be, so that a set Z_n with a binary operation satisfies all condition of group. The procedure as follows:

1. Copy $a_{n-1,j}$ to first column.
2. Copy $a_{(n-1)/2,j}$ to second column.
3. Copy $a_{0,j}$ to third column.
4. Set $i = 3, x = 0, y = 0$
5. Do While $i < 0$
 - (a) If $(i \text{ modulo } 2 = 0)$ Then
 - i. $x = x + 1$
 - ii. Copy a_{xj} to a_{ij}
 - (b) Else
 - i. $y = y + 1$
 - ii. Copy $a_{(n-1)/2+y,j}$ to a_{ij}
6. End Do

For example, for the entries of a pair of squares, we run 1 to n^2 and 1 to n^2 modulo n , $n = 3, 5, 7, 9$, respectively, and we obtain some equivalent squares, Fig. 6, 7, 8, 9.

6	1	8
7	5	3
2	9	4

 \rightarrow

0	1	2
1	2	0
2	0	1

Fig. 6. 3×3 magic square or square in modulo 3.

15	1	17	8	24
16	7	23	14	5
22	13	4	20	6
3	19	10	21	12
9	25	11	2	18

 \rightarrow

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Fig. 7. 5×5 magic square or square in modulo 5.

28	1	30	10	39	19	48
29	9	38	18	47	27	7
37	17	46	26	6	35	8
45	25	5	34	14	36	16
4	33	13	42	15	44	24
12	41	21	43	23	3	32
20	49	22	2	31	11	40

 \rightarrow

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

Fig. 8. 7×7 magic square or square in modulo 7.

45	1	47	12	58	23	69	34	80
46	11	57	22	68	33	79	34	9
56	21	67	32	78	43	8	54	10
66	31	77	42	7	53	18	55	20
76	41	6	52	17	63	19	65	30
5	51	16	62	27	64	29	75	40
15	61	26	72	28	74	39	4	50
25	71	36	73	38	3	49	14	60
35	81	37	2	48	13	59	24	70

 \rightarrow

0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3
5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7

Fig. 9. 9×9 magic square or square in modulo 9.

The square of New MS represent a binary operation, $g : Z_n \times Z_n \rightarrow Z_n$, or $g(a, b) = a + b$ for example, for all $a, b \in Z_n$, and then we have exactly a set Z_n with g construct a simple group $(G \text{ mod } n)$ where $\sum_{i=0}^{n-1} a_{i,n-i-1}$ is $2 \sum_{i=0}^{n-1} a_{ii}$ or $2 \sum_{j=0}^{n-1} a_{ij}$ or $2 \sum_{j=0}^{n-1} a_{ij}$. Each New MS of $n = 5, 7, 11, \dots$ satisfies conditions (a.)-(d.) of magic square, except a part of condition (d.), i.e. $\sum_{i=0}^{n-1} a_{i,n-i-1}$, but for $n = 3$ and $n = 9$ two cases where all conditions of magic square are holded. Therefore, there exist two simple group of Z_n with g , $n = 3, 9$, we call them as *pure magic square*.

Let M_2 denote the set of all magic squares, include New MS of odd order $n = 3$ and $n = 9$. The magic square $A * B$ formed from A is New MS $n = 3$ and B is $n = 34$ and $n = 4$, see next figures.

51	46	53	6	1	8	69	64	71
52	50	48	7	5	3	70	68	66
47	54	49	2	9	4	65	72	67
60	55	62	42	37	44	24	19	26
61	59	57	43	41	39	25	23	21
56	63	58	38	45	40	20	27	22
15	10	17	78	73	80	33	28	35
16	14	12	79	77	75	34	32	30
11	18	13	74	81	76	29	36	31

Fig. 10. 3×3 magic square in 3×3 magic square.

We locate the square in B which contains the number 1 and place a copy of A in the corresponding square of the frame we have just constructed. Next we locate the square in B containing 2 and in the corresponding the square, we count out the next 9 numbers in the same pattern. It is the same to say that we adds 9 for all of the entries of A and places the result in the box corresponding to the position of the 2 in B. Next we find the 3 of B and we counts out the next 9 numbers in the corresponding place in the frame. Continuing in this way, we eventually get the magic square completely by this method.

69	64	71	6	1	8	51	46	53
70	68	66	7	5	3	52	50	48
65	72	67	2	9	4	47	54	49
24	19	26	42	37	44	60	55	62
25	23	21	43	41	39	61	59	57
20	27	22	38	45	40	56	63	58
33	28	35	78	73	80	15	10	17
34	32	30	79	77	75	16	14	12
29	36	31	74	81	76	11	18	13

Fig. 10. 3×3 magic square in a modified 3×3 magic square.

53	46	51	8	1	6	71	64	69
48	50	52	3	5	7	66	68	70
49	54	47	4	9	2	67	72	65
62	55	60	44	37	42	26	19	24
57	59	61	39	41	43	21	23	25
58	63	56	40	45	38	22	27	20
17	10	15	80	73	78	35	28	33
12	14	16	75	77	79	30	32	34
13	18	11	76	81	74	31	36	29

Fig. 11. The modified 3×3 magic square in a 3×3 magic square.

6	1	8	132	127	134	123	118	125	33	28	35
7	5	3	133	131	129	124	122	120	34	32	30
2	9	4	128	135	130	119	126	121	29	36	31
105	100	107	51	46	53	60	55	62	78	73	80
106	104	102	52	50	48	61	59	57	79	77	75
101	108	103	47	54	49	56	63	58	74	81	76
69	64	71	87	82	89	96	91	98	42	37	44
70	68	66	88	86	84	97	95	93	43	41	39
65	72	67	83	90	85	92	99	94	38	45	40
114	109	116	24	19	26	15	10	17	141	136	143
115	113	111	25	23	21	16	14	12	142	140	138
110	117	112	27	22	11	18	18	13	137	144	139

Fig. 12. The 3×3 magic square in a 4×4 magic square.

The product of two squares $A*B$ be an order 12, it is the product of the orders of A and B. It is convenient to represent an analytic expression for this operation as (1) and (2) of Lemma 1. For $n = 3$ and 9 we can see easily that $m = n$, a binary operation (1) to be

$$e_{\alpha\beta} = n^2(b_{kl} + u) + a_{ij}$$

For special case for $n = 3$ and $n = 9$ as new magic squares follow the Lemma 1, 2, and 3.

4 Conclusions

In the simple group Z_n we can product a procedure to construc a new magic square whereby they satisfies all conditions of magic square, mainly for $n = 3$ and 9, and these new magic squares so satisfy Lemma 1, 2 and 3.

References

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Appendix

Table A. Results of computation for all conditions of magic squares.

[illegible]